Let $Y_1, \dots, Y_n$ be a Fundamental set of solutions of $Y' = P(t)\,Y$

(Fundamental Matrix) $\Phi = \begin{bmatrix} Y_1 & Y_2 & \cdots & Y_n \end{bmatrix}_{n \times n}$

$(W = |\Phi| \neq 0)$

Show $\boxed{\Phi' = P(t) \cdot \Phi}$

: Since $Y_1, \dots, Y_n$ are Fundamental solutions $Y_1' = PY_1,\ Y_2' = PY_2 \ \dots\ Y_n' = PY_n$

$\Phi' = \begin{bmatrix} Y_1 & Y_2 & \cdots & Y_n \end{bmatrix}'_{n \times n}$

$= \begin{bmatrix} Y_1' & Y_2' & \cdots & Y_n' \end{bmatrix}_{n \times n}$

$= \begin{bmatrix} PY_1 & PY_2 & \cdots & PY_n \end{bmatrix}_{n \times n}$

$= P \begin{bmatrix} Y_1 & Y_2 & \cdots & Y_n \end{bmatrix}$

I-Sect 4.8 Variation of Parameters Method

Sec 4.8: Nonhomogeneous 1st Order Linear Systems.

Has the form standard form

$$\vec{Y}\,' = P(t)\cdot\vec{Y} + \vec{G}(t), \quad \vec{Y}(t_0) = Y_0, \qquad a < t < b$$

Example:

$$\vec{Y}\,' = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}\vec{Y} + \begin{bmatrix} e^t \\ 0 \end{bmatrix}, \quad \vec{Y}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

How to solve this type of system?
Variation of Parameters Method: First solve the homogeneous problem

$$\vec{Y}\,' = P(t)\cdot\vec{Y}$$

Let $\vec{Y_c}(t)$ be the homogeneous solution; i.e., $\vec{Y_c}(t) = \Phi(t)\mathbf{c}$ where $\Phi(t)$ is a fundamental matrix for the homogeneous equation and $\mathbf{c}$ is an arbitrary $n\times 1$ column vector.

Now suppose that there is a particular solution for the nonhomogeneous equation that has the form

$$\vec{Y_p}(t) = \Phi(t)\mathbf{u}(t)$$

where $\mathbf{u}(t)$ is an $n\times 1$ column vector. Then, $\vec{Y_p}\,' = P(t)\cdot\vec{Y_p} + G(t)$ and so

$$\Phi'(t)\mathbf{u}(t) + \Phi(t)\mathbf{u}'(t) = \mathbf{P}(t)\Phi(t)\mathbf{u}(t) + \vec{G}(t).$$

Since $\Phi(t)$ is a fundamental matrix for the homogeneous system, we have $\Phi'(t) = P(t)\cdot\Phi(t)$.This yields

$$\Phi(t)\mathbf{u}'(t) = \vec{G}(t).$$

Hence the column vector $\mathbf{u}(t)$ must be $\int [\Phi(t)]^{-1}\cdot\vec{G}(t)\,dt$.

① $Y_c = \Phi\, c$
② $Y_p = \Phi\cdot u$
③ Plug Y_p into $Y' = PY + G$
$(\Phi u)' = P(\Phi u) + G$
(Product rule) $\Phi' u + \Phi u' = P\cdot\Phi u + G$
$(P\cdot\Phi) u + \Phi u' = P\cdot\Phi u + G$
$\Phi u' = G$
$u' = \Phi^{-1} G$ ($|\Phi| \neq 0$, thus Φ^{-1} exists)
$u = \int \Phi^{-1} G\, dt$
DNE

Under the above assumptions we have the following algorithm:

- The non homogeneous system must be given in standard form.
- Identify the matrix $P(t)$ and the column vector $\vec{G}(t)$.
- Find a fundamental matrix $\Phi(t)$ for the homogeneous system $\vec{Y}\,' = P(t)\cdot\vec{Y}$.
- Compute the inverse of $\Phi(t)$.
- Compute the product $[\Phi(t)]^{-1}\cdot\vec{G}(t)$.
- Set $\mathbf{u}(t) = \int [\Phi(t)]^{-1}\cdot\vec{G}(t)\,dt$.
- Set $\vec{Y_p}(t) = \Phi(t)\mathbf{u}(t)$.
- The general solution to the nonhomogeneous problem is given by $\vec{Y}(t) = \vec{Y_c}(t) + \vec{Y_p}(t)$.

- Set $Y_p(t) = \Phi(t)\mathbf{u}(t)$.
- The general solution to the nonhomogeneous problem is given by $\vec{Y}(t) = \vec{Y_c}(t) + \vec{Y_p}(t)$.
- Finally, use the initial conditions to get the solution to the i.v.p.

No Restriction

Ex 1: Solve the i.v.p.

$$\vec{Y}\,' = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \vec{Y} + \begin{bmatrix} e^t \\ 0 \end{bmatrix}, \quad \vec{Y}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

(By variation of parameters)

① Y_c: soln of $Y' = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} Y$

Eigenpair: $(\lambda_1 = -1, \begin{bmatrix} 1 \\ -1 \end{bmatrix}), (\lambda_2 = 3, \begin{bmatrix} 1 \\ 1 \end{bmatrix})$

Two soln: $Y_1 = e^{-t}\begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad Y_2 = e^{3t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$Y_c = C_1 Y_1 + C_2 Y_2$

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1}$

$A^{-1} = \frac{1}{|A|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

② Φ, Φ^{-1}

- $\Phi = \begin{bmatrix} e^{-t} & e^{3t} \\ -e^{-t} & e^{3t} \end{bmatrix}$

- $\Phi^{-1} = \frac{1}{|\Phi|}\begin{bmatrix} e^{3t} & -e^{3t} \\ e^{-t} & e^{-t} \end{bmatrix}$ $\quad |\Phi| = e^{2t} + e^{2t} = 2e^{2t}$

$= \frac{1}{2e^{2t}}\begin{bmatrix} e^{3t} & -e^{3t} \\ e^{-t} & e^{-t} \end{bmatrix}$ $\quad \frac{1}{e^{2t}} = e^{-2t}$

$= \frac{1}{2}\begin{bmatrix} e^{t} & -e^{t} \\ e^{-3t} & e^{-3t} \end{bmatrix}$

③ $= \frac{1}{2}\int \begin{bmatrix} e^{t} & -e^{t} \\ e^{-3t} & e^{-3t} \end{bmatrix}\begin{bmatrix} e^t \\ 0 \end{bmatrix} dt$ $\quad G = \begin{bmatrix} e^t \\ 0 \end{bmatrix}$

$= \frac{1}{2}\int \begin{bmatrix} e^{2t} \\ e^{-2t} \end{bmatrix} dt$

$= \frac{1}{2}\begin{bmatrix} \int e^{2t}\,dt \\ \int e^{-2t}\,dt \end{bmatrix} = \frac{1}{2}\begin{bmatrix} \frac{1}{2}e^{2t} \\ -\frac{1}{2}e^{-2t} \end{bmatrix} = \frac{1}{4}\begin{bmatrix} e^{2t} \\ -e^{-2t} \end{bmatrix}$ "u"

④ $Y_p = \Phi \cdot u$

$$= \frac{1}{4}\begin{bmatrix} e^{-t} & e^{3t} \\ -e^{-t} & e^{3t} \end{bmatrix}\begin{bmatrix} e^{2t} \\ -e^{-2t} \end{bmatrix} = \frac{1}{4}\begin{bmatrix} e^{t} - e^{t} = 0 \\ -e^{t} - e^{t} = -2e^{t} \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ -\frac{1}{2}e^{t} \end{bmatrix}$$

⑤ General Sol$_n$

$$Y = Y_c + Y_p = C_1\begin{bmatrix} e^{-t} \\ -e^{-t} \end{bmatrix} + C_2\begin{bmatrix} e^{3t} \\ e^{3t} \end{bmatrix} + \begin{bmatrix} 0 \\ -\frac{1}{2}e^{t} \end{bmatrix}$$

⑥ $C_1, C_2 = ?$ $\quad Y(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$Y(0) = C_1\begin{bmatrix} 1 \\ -1 \end{bmatrix} + C_2\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} C_1 + C_2 \\ -C_1 + C_2 - \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$C_1 + C_2 = 1$$
$$+ \quad -C_1 + C_2 = \frac{1}{2}$$
$$2C_2 = \frac{3}{2}$$
$$C_2 = \frac{3}{4}$$
$$C_1 = \frac{1}{4}$$

$$Y(t) = \frac{1}{4}\begin{bmatrix} e^{-t} \\ -e^{-t} \end{bmatrix} + \frac{1}{4}\begin{bmatrix} e^{3t} \\ e^{3t} \end{bmatrix} + \begin{bmatrix} 0 \\ -\frac{1}{2}e^{t} \end{bmatrix}$$